MAA AMC
American Mathematics Competitions

Solutions Pamphlet

MAA American Mathematics Competitions

20th Annual

AMC 10A

American Mathematics Competition 10A

Thursday, February 7, 2019

This Pamphlet gives at least one solution for each problem on this year's competition and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic versus geometric, computational versus conceptual, elementary versus advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *Copies of the problem booklet and solution pamphlet may be shared with your students for educational purposes. However, the publication, reproduction, or communication of the problems or solutions for this competition with anyone outside of the classroom is a violation of the competition rules. This includes dissemination via copier, telephone, email, internet, or media of any type.*

Correspondence about the problems/solutions for this AMC 10 and orders for any publications should be addressed to:

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The problems and solutions for this AMC 10 were prepared by
MAA's Subcommittee on the AMC 10/AMC 12 Exams, under the direction of the
co-chairs Jerrold W. Grossman and Carl Yerger.

1. **Answer (C):**

$$2^{\left(0^{\left(1^9\right)}\right)} + \left(\left(2^0\right)^1\right)^9 = 2^{\left(0^1\right)} + \left((1)^1\right)^9 = 2^0 + 1^9 = 1 + 1 = 2$$

2. **Answer (A):** Both 20! and 15! have at least 3 factors of 2 and at least 3 factors of 5, so both are multiples of $10^3 = 1000$ and therefore have 0s as their last three digits. The digit in the hundreds place of the difference is therefore $0 - 0 = 0$.

3. **Answer (D):** If B denotes Bonita's age last year, then Ana's age last year was $5B$. This year Bonita's age is $B + 1$, and Ana's age is $5B + 1$. So the given condition is $5B + 1 = (B + 1)^2$. This quadratic equation simplifies to $B^2 = 3B$. Because the two girls were born in different years, $B \neq 0$, so $B = 3$. Last year Bonita was 3 and Ana was $5 \cdot 3 = 15$, so they were born $15 - 3 = 12$ years apart. As a check, note that Ana's age this year, 16, is indeed the square of Bonita's age this year, 4.

4. **Answer (B):** The greatest number of balls that can be drawn without getting 15 of one color is obtained if and only if 14 red, 14 green, 14 yellow, 13 blue, 11 white, and 9 black balls are drawn, a total of 75 balls. When another ball is drawn, it will be the 15th ball of one of the colors red, green, or yellow. Thus the requested minimum is $75 + 1 = 76$.

5. **Answer (D):** The sum $(-44) + (-43) + \cdots + 43 + 44 + 45 = 45$. This sum has 90 consecutive integers. There is no longer list because for the sum of consecutive integers to be positive, there must be more positive integers than negative integers. Further, if there are more than 90 consecutive integers as part of a list that sums to a positive number, then there must be a positive integer greater than 45 that is not cancelled out by its additive inverse.

OR

Suppose that the consecutive integers are $a, a+1, \ldots, a+n-1$; their sum then equals $na + \frac{n(n-1)}{2}$. Therefore

$$na + \frac{n(n-1)}{2} = 45,$$

so $n(2a+n-1) = 90$, which implies that $n \leq 90$. A sequence of 90 consecutive integers with sum equal to 45 indeed exists, as observed in the first solution.

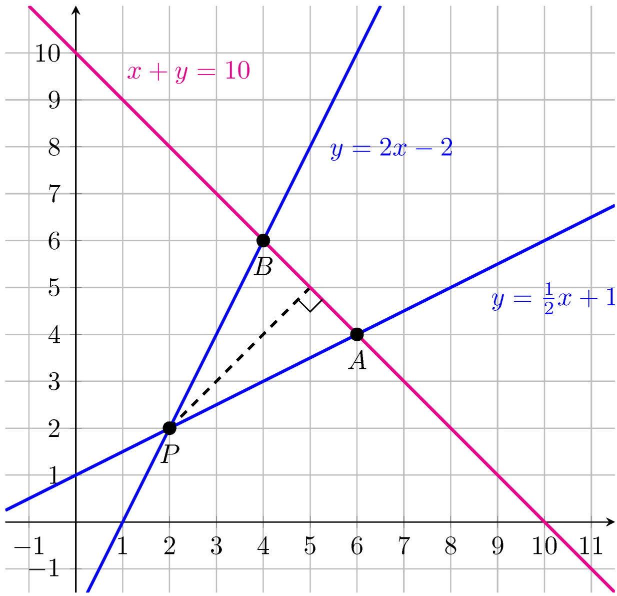
6. **Answer (C):** In a square or non-square rectangle, the diagonals are congruent and bisect each other, and their point of intersection is equidistant from all four vertices. This point also lies on the perpendicular bisectors of all four sides. In a non-square rhombus or a parallelogram that is not a rectangle or rhombus, the perpendicular bisectors of parallel sides do not meet, so no point could be equidistant from all four vertices. Finally, in an isosceles trapezoid that is not a parallelogram, the perpendicular bisectors of the parallel sides are the same line, and the perpendicular bisectors of the nonparallel sides meet at a point on this line; that point is equidistant from all four vertices. In summary, 3 of the types of given quadrilaterals—the first, second, and fifth in the list—have the required property.

OR

The required point exists if and only if the quadrilateral can be inscribed in a circle, in which case the point is the center of the circle. A square, rectangle, and isosceles trapezoid can each be inscribed in a circle, but a non-square rhombus and a non-rectangular parallelogram cannot. Therefore the required point exists for 3 of the listed types of quadrilaterals.

7. **Answer (C):** Let $P(2,2)$ be the intersection point. The two lines have equations $y = \frac{1}{2}x+1$ and $y = 2x-2$. They intersect $x+y = 10$ at $A(6,4)$ and $B(4,6)$. Consider $\overline{AB}$ to be the base of the triangle; then the altitude of the triangle is the segment joining $(2,2)$ and $(5,5)$. By the Distance Formula, the area of $\triangle PAB$ is

$$\frac{1}{2} \cdot \sqrt{(6-4)^2+(4-6)^2} \cdot \sqrt{(5-2)^2+(5-2)^2} = \frac{1}{2} \cdot 2\sqrt{2} \cdot 3\sqrt{2} = 6.$$

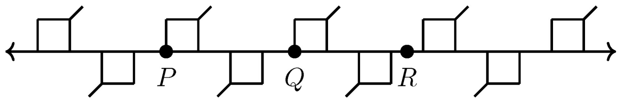


Note: The area of the triangle with vertices $(2,2)$, $(6,4)$, and $(4,6)$ can be calculated in a number of other ways, such as by enclosing it in a 4×4 square with sides parallel to the coordinate axes and subtracting the areas of three right triangles; by splitting it into two triangles with the line $y = 4$; by the shoelace formula:

$$\frac{1}{2} \cdot \Big|(2 \cdot 4 + 6 \cdot 6 + 4 \cdot 2) - (6 \cdot 2 + 4 \cdot 4 + 2 \cdot 6)\Big| = 6;$$

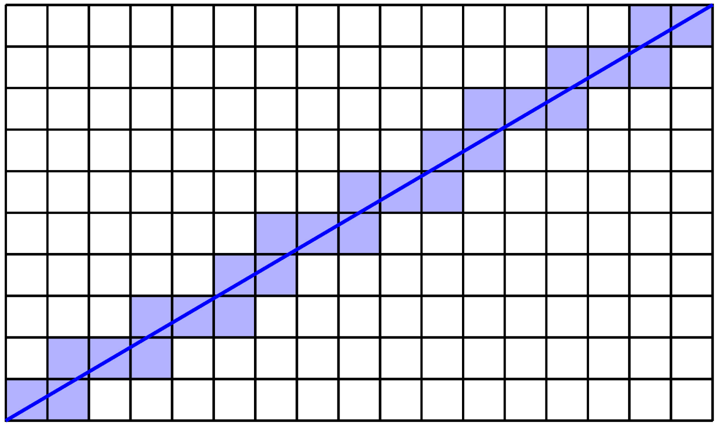
or by observing that there are 4 lattice points in the interior of the triangle and 6 lattice points on the boundary, and using Pick's Formula: $4 + \frac{6}{2} - 1 = 6$.

8. **Answer (C):** A translation in the direction parallel to line ℓ by an amount equal to the distance between the left sides of successive squares above the line (or any integer multiple thereof), will take the figure to itself. The translation vector could be $\overrightarrow{PQ}$ in the figure below. In addition, a rotation of 180° around any point on line ℓ that is halfway between the bases on ℓ of a square above the line and a nearest square below the line, such as point R in the figure, will also take the figure to itself. Either of the given reflections, however, will result in a figure in which the "tails" attached to the squares above the line are on the left side of the squares instead of the right side. Therefore 2 of the listed non-identity transformations will transform this figure into itself.



9. **Answer (B):** The sum of the first n positive integers is $\frac{n(n+1)}{2}$, and the product of the first n positive integers is $n! = n \cdot (n-1) \cdots 2 \cdot 1$. If $n > 1$ is odd, then $n \cdot \frac{n+1}{2}$ divides $n \cdot (n-1)!$ because $\frac{n+1}{2}$ is an integer between 1 and n. If n is even, then $\frac{n}{2} \cdot (n+1)$ does not divide $n!$ if and only if $n+1$ is prime. Because 997 is the greatest three-digit prime number, the greatest three-digit positive integer n for which the sum of the first n positive integers is not a divisor of the product of the first n positive integers is $997 - 1 = 996$.

10. **Answer (C):** Because 10 and 17 are relatively prime, the diagonal does not cross the boundaries between tiles at any corner point of the tiles. In order for the bug to move from one vertex of the rectangle to the opposite vertex, the bug must cross 9 edges in one direction and 16 edges in the other direction, a total of 25 edges. Each time the bug crosses an edge, it enters a new tile. Counting the tile it started on as well, the bug visits a total of $1 + 25 = 26$ tiles.

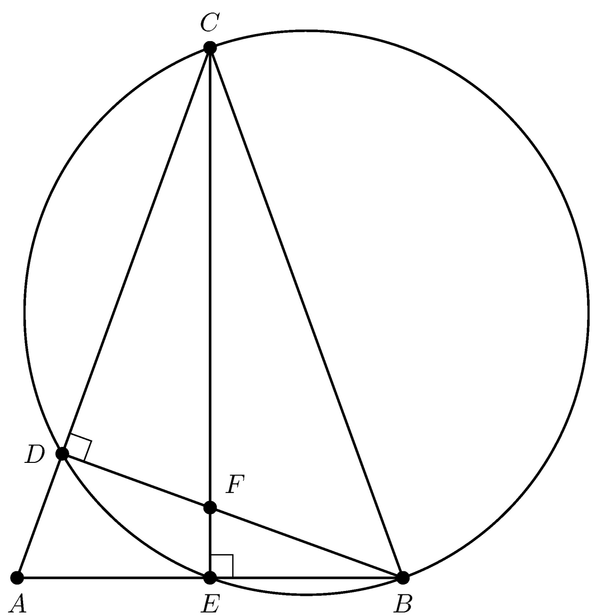


11. **Answer (C):** Because $201^9 = 3^9 \cdot 67^9$, a square divisor has the form $3^a \cdot 67^b$ where $a, b \in \{0, 2, 4, 6, 8\}$, and a cubic divisor has the form $3^a \cdot 67^b$ where $a, b \in \{0, 3, 6, 9\}$. A number is both a square and a cube if and only if it is a sixth power, so it has the form $3^a \cdot 67^b$ where $a, b \in \{0, 6\}$. Thus there are $5 \cdot 5 = 25$ square divisors, $4 \cdot 4 = 16$ cubic divisors, and $2 \cdot 2 = 4$ divisors that are sixth powers. Therefore the number of divisors that are squares and/or cubes is $25 + 16 - 4 = 37$.

12. **Answer (E):** Each of the values 1 through 28 is a mode, so $d = \frac{14+15}{2} = 14.5$. There are $15 \cdot 12 = 180 < \frac{365}{2}$ data values less than or equal to 15, and there are $16 \cdot 12 = 192 > \frac{365}{2}$ values less than or equal to 16. Therefore more than half of the values are greater than or equal to 16 and more than half of the values are less than or equal to 16, so $M = 16$. To see the relationship between μ and 16, note that if every month had 31 days, then there would be 12 of each value from 1 to 31, and the mean would be 16; because the actual data are missing some of the larger values, $\mu < 16$. To see the relationship between μ and 14.5, note that if every month had 28 days, then there would be 12 of each value from 1 to 28, and the mean would be 14.5; because the actual data consist of all of these values together with some larger values, $\mu > 14.5$. Therefore $d = 14.5 < \mu < 16 = M$.

13. **Answer (D):** Because $BC = AC$ and $\angle ACB = 40^\circ$, it follows that $\angle BAC = \angle ABC = 70^\circ$. Because $\angle BAC = \frac{1}{2}\left(\widehat{BC} - \widehat{DE}\right)$ and $\widehat{BC} = 180^\circ$, it follows that $\widehat{DE} = 40^\circ$. Then

$$\angle BFC = \frac{1}{2}\left(\widehat{BC} + \widehat{DE}\right) = \frac{1}{2}\left(180^\circ + 40^\circ\right) = 110^\circ.$$



OR

Because D and E lie on the circle with diameter $\overline{BC}$, both $\angle BDC$ and $\angle BEC$ are right angles, so $\angle ADF$ and $\angle AEF$ are also right angles. Therefore in quadrilateral $AEFD$

$$\angle DFE = 180^\circ - \angle DAE = 180^\circ - \frac{1}{2}(180^\circ - \angle ACB) = 110^\circ,$$

and $\angle BFC$ has the same measure.

14. **Answer (D):** There are several cases to consider.

If all four lines are concurrent, then there is 1 intersection point.

If three of the lines are concurrent and the fourth line is parallel to one of those three, then there are 3 intersection points. If three of the lines are concurrent and the fourth line is parallel to none of those three, then there are 4 intersection points.

In the remaining cases no three lines are concurrent. If they are all parallel, then there are 0 intersection points.

If only three of them are parallel, then there are again 3 intersection points.

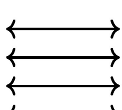
If two of them are parallel but no three are mutually parallel, then there are either again 4 intersection points, if the other two lines are parallel to each other; or 5 intersection points, if the other two lines intersect.

In the final case, every line intersects every other line, giving 6 points of intersection.

These are all the cases, so the requested sum is $1+3+4+0+5+6 = 19$.

OR

The problem deals with the possible number of intersection points when four distinct lines are drawn in the plane. The maximum number of intersection points is $\binom{4}{2} = 6$. The diagram shows that the number of intersection points can be 0, 1, 3, 4, 5, and 6.



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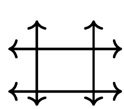

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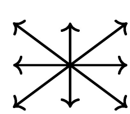

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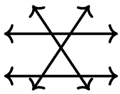

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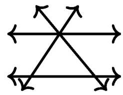

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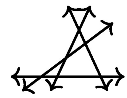


6

To see that there cannot be 2 points of intersection, suppose to the contrary that A and B are the only points of intersection among the four lines. If A and B lie on one of the lines, then the other three lines, ℓ, m, and n, must all pass through A or B, with at least one line passing through each. Without loss of generality, suppose ℓ contains A, and m and n contain B. At most one of m and n can be parallel to ℓ, and thus there will be a third intersection point. Otherwise it must be the case that A is the intersection of two of the lines, and B is the intersection of the other two lines; say A is the intersection of k and ℓ, and B is the intersection of m and n. Again, m and n cannot both be parallel to ℓ, and thus there must be another point of intersection.

As in the first solution, the requested sum is $0+1+3+4+5+6 = 19$.

Note: Students who wish to pursue further interesting questions about arrangements of points and lines, such as the dual problem to this one (how many lines n points can determine), might want to start by looking up the Sylvester–Gallai Theorem.

15. **Answer (E):** The sequence begins $1, \frac{3}{7}, \frac{3}{11}, \frac{3}{15}, \frac{3}{19}, \ldots$. This pattern leads to the conjecture that $a_n = \frac{3}{4n-1}$. Checking the initial conditions $n = 1$ and $n = 2$, and observing that for $n \geq 3$,

$$\begin{aligned}\frac{\frac{3}{4(n-2)-1}\cdot\frac{3}{4(n-1)-1}}{2\cdot\frac{3}{4(n-2)-1}-\frac{3}{4(n-1)-1}} &= \frac{\frac{3}{4n-9}\cdot\frac{3}{4n-5}}{\frac{6}{4n-9}-\frac{3}{4n-5}}\\ &= \frac{9}{6(4n-5)-3(4n-9)}\\ &= \frac{9}{12n-3} = \frac{3}{4n-1},\end{aligned}$$

confirms the conjecture. Therefore $a_{2019} = \frac{3}{4\cdot 2019-1} = \frac{3}{8075}$, and the requested sum is $3 + 8075 = 8078$.

OR

Taking the reciprocal of both sides of the recurrence gives

$$\frac{1}{a_n} = \frac{2a_{n-2}-a_{n-1}}{a_{n-2}\cdot a_{n-1}} = \frac{2}{a_{n-1}} - \frac{1}{a_{n-2}},$$

which is equivalent to

$$\frac{1}{a_n} - \frac{1}{a_{n-1}} = \frac{1}{a_{n-1}} - \frac{1}{a_{n-2}}.$$

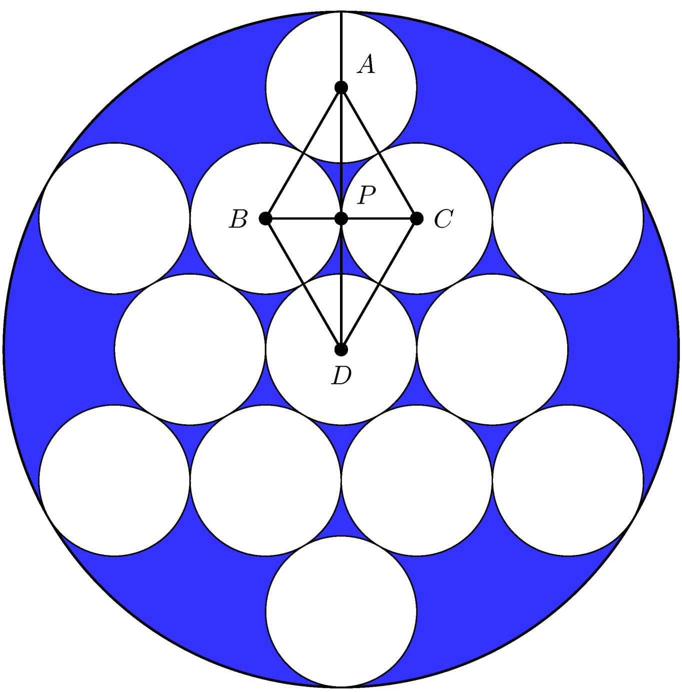
Thus the sequence of reciprocals is an arithmetic sequence. Its first term is 1, and its common difference is $\frac{7}{3} - 1 = \frac{4}{3}$. Its 2019th term is

$$\frac{1}{a_{2019}} = 1 + 2018 \cdot \frac{4}{3} = \frac{8075}{3},$$

so $a_{2019} = \frac{3}{8075}$, and the requested sum is $3 + 8075 = 8078$.

16. **Answer (A):** Let A, B, C, and D be the centers of four of the circles as shown below, and let P be the intersection of the diagonals of rhombus $ABDC$. Then $PC = 1$ and $AC = 2$, so $AP = \sqrt{3}$; similarly $PD = \sqrt{3}$. The radius of the large circle is therefore $1 + 2\sqrt{3}$. The requested area is

$$\pi\left(1 + 2\sqrt{3}\right)^2 - 13\pi = 4\pi\sqrt{3}.$$



Note: This problem is related to the question of how densely the plane can be packed with congruent circles—how much wasted space there is with the most efficient packing. It has been proved that the best arrangement is the one shown in this problem, with each circle surrounded by six others. The fraction of the plane covered by the circles is $\frac{\pi}{6}\sqrt{3} \approx 0.9069$.

17. **Answer (D):** Of the 9 cubes available, 1 cube will not be used. Because there are three different kinds of cubes and limited numbers of each kind, there are three different possibilities for the set of cubes that are used. One possibility is 1 red cube, 3 blue cubes, and 4 green cubes; the second possibility is 2 red cubes, 2 blue cubes, and 4 green cubes; and the third possibility is 2 red cubes, 3 blue cubes, and 3 green cubes. Cubes of the same color are indistinguishable. Hence the number of different towers is

$$\frac{8!}{1!\cdot 3!\cdot 4!}+\frac{8!}{2!\cdot 2!\cdot 4!}+\frac{8!}{2!\cdot 3!\cdot 3!}=1{,}260.$$

OR

There is a one-to-one correspondence between towers of height 9 and towers of height 8 by viewing the top cube in a tower of height 9 as the cube that is not used in a tower of height 8. The number of different towers of height 9 is given by

$$\frac{9!}{2!\cdot 3!\cdot 4!}=1{,}260.$$

Note: A generalization of the two solutions put together yields the following extension of Pascal's Identity:

$$\frac{(a+b+c-1)!}{(a-1)!\cdot b!\cdot c!}+\frac{(a+b+c-1)!}{a!\cdot (b-1)!\cdot c!}+\frac{(a+b+c-1)!}{a!\cdot b!\cdot (c-1)!}=\frac{(a+b+c)!}{a!\cdot b!\cdot c!}.$$

18. **Answer (D):** The number $0.\overline{23}_k$ is the sum of an infinite geometric series with first term $\frac{2}{k}+\frac{3}{k^2}$ and common ratio $\frac{1}{k^2}$. Therefore the sum is

$$\frac{\frac{2}{k}+\frac{3}{k^2}}{1-\frac{1}{k^2}}=\frac{2k+3}{k^2-1}=\frac{7}{51}.$$

Then $0=7k^2-102k-160=(k-16)(7k+10)$, and therefore $k=16$.

OR

Let $x = 0.\overline{23}_k$. Then $(k^2-1)x = 23.\overline{0}_k = 2k+3$, so $\frac{2k+3}{k^2-1} = \frac{7}{51}$ and the solution proceeds as above.

Note: If $0 < a < q$, $\gcd(a,q) = 1$, and $\gcd(k,q) = 1$, then the base-k representation of the fraction $\frac{a}{q}$ has least period equal to the order of k modulo q. In the case at hand, $k = 16$ and $q = 51 = 3 \cdot 17$. Then $\phi(q) = 2 \cdot 16 = 32$ and $k^2 = 256 = 5 \cdot 51 + 1$, so k has order 2 modulo 51.

19. **Answer (B):** Observe that

$$\begin{aligned}
&(x+1)(x+4)(x+2)(x+3)+2019\\
&\qquad= \left(x^2+5x+4\right)\left(x^2+5x+6\right)+2019\\
&\qquad= \left[\left(x^2+5x+5\right)-1\right]\left[\left(x^2+5x+5\right)+1\right]+2019\\
&\qquad= \left(x^2+5x+5\right)^2-1+2019\\
&\qquad= \left(x^2+5x+5\right)^2+2018.
\end{aligned}$$

Because $(x^2+5x+5)^2 \geq 0$ for all x and equals 0 for $x = \frac{-5\pm\sqrt{5}}{2}$, it follows that the requested minimum value is 2018.

OR

Let $r = x + \frac{5}{2}$. Then

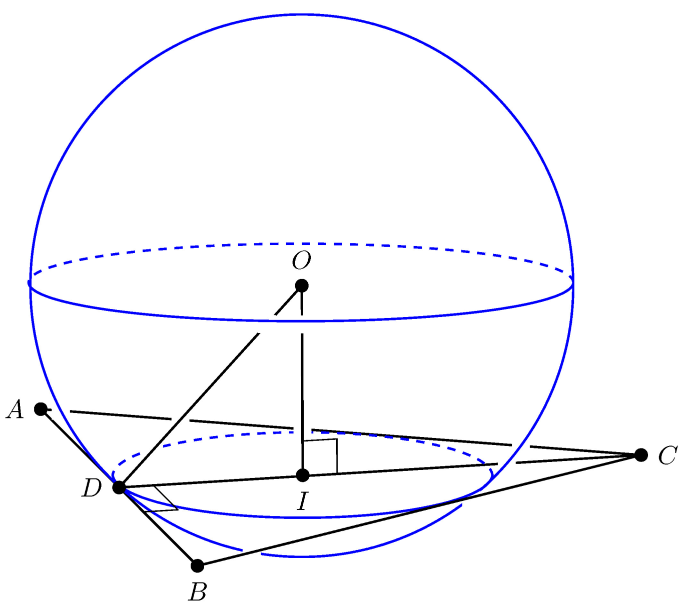
$$\begin{aligned}
(x+1)(x+2)(x+3)(x+4) &= \left(r-\frac{3}{2}\right)\left(r-\frac{1}{2}\right)\left(r+\frac{1}{2}\right)\left(r+\frac{3}{2}\right)\\
&= \left(r^2-\frac{1}{4}\right)\left(r^2-\frac{9}{4}\right)\\
&= \left(r^2-\frac{5}{4}\right)^2-1,
\end{aligned}$$

the minimum value of which is -1. Therefore the minimum value of the given expression is $2019-1=2018$.

20. **Answer (B):** The sum of three integers is odd exactly when either all of the integers are odd, or one is odd and two are even. Five of the numbers $1, 2, \ldots, 9$ are odd, so at least one row must contain two or more odd numbers. Thus one row must contain three odd

numbers and no even numbers, and the other two rows must contain one odd number and two even numbers. The same is true of the three columns. There are $3 \times 3 = 9$ ways to choose which row and which column contain all odd numbers, and then the remaining four squares must have even numbers. There are $\binom{9}{4} = 126$ ways in total to choose which squares have odd numbers and which have even numbers, so the desired probability is $\frac{9}{126} = \frac{1}{14}$.

21. **Answer (D):** Let $\triangle ABC$ be the given triangle, with $AB = 24$ and $AC = BC = 15$, and let D be the midpoint of $\overline{AB}$. The length of the altitude to $\overline{AB}$ is $CD = \sqrt{15^2 - 12^2} = 9$. The area of $\triangle ABC$ is $\frac{1}{2} \cdot 24 \cdot 9 = 108$. The plane of the triangle intersects the sphere in a circle, which is the inscribed circle for $\triangle ABC$. Let r and I be the radius and the center of the inscribed circle, respectively. The semiperimeter of the triangle is $\frac{1}{2}(AB + BC + AC) = 27$, so $r = \frac{108}{27} = 4$. In right triangle DIO the hypotenuse $\overline{OD}$ has length 6 (the radius of the sphere) and $DI = r = 4$, so $OI = \sqrt{36 - 16} = 2\sqrt{5}$, the requested distance between the center of the sphere and the plane determined by $\triangle ABC$.

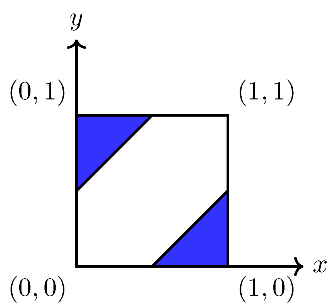


22. **Answer (B):** The probability that the first coin flip for both x and y is heads is $\frac{1}{4}$, and in half of these cases $|x - y|$ will be 0 and in the

other half of these cases $|x-y|$ will be 1. This contributes $\frac{1}{4}\cdot\frac{1}{2}=\frac{1}{8}$ to the probability that $|x-y|>\frac{1}{2}$.

The probability that the first coin flip for x is heads and the first coin flip for y is tails or vice versa is $\frac{1}{2}$. In such cases, one of the variables is 0 or 1, and the probability that $|x-y|>\frac{1}{2}$ is $\frac{1}{2}$. This contributes $\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}$ to the probability that $|x-y|>\frac{1}{2}$.

Finally, $\frac{1}{4}$ of the time both x and y will be chosen uniformly from $[0,1]$. In this case, the situation can be modeled by the following diagram, in which the area of the shaded region gives the probability that $|x-y|>\frac{1}{2}$. This contributes $\frac{1}{4}\cdot\frac{1}{4}=\frac{1}{16}$ to the probability that $|x-y|>\frac{1}{2}$. The requested probability is $\frac{1}{8}+\frac{1}{4}+\frac{1}{16}=\frac{7}{16}$.



23. **Answer (C):** Define a round to be the recitations done during the successive turns of Tadd, Todd, and Tucker, in that order. Note that Tadd says 1 number in round 1, 4 numbers in round 2, 7 numbers in round 3, 10 numbers in round 4, and, in general, $3N-2$ numbers in round N. In turn, Todd says $3N-1$ numbers and Tucker says $3N$ numbers in round N. Therefore $9N-3$ numbers are recited by all three children in round N. During the course of the first N rounds Tadd recites a total of $1+4+7+10+\cdots+(3N-2)$ numbers. The sum of this arithmetic series is

$$N\cdot\frac{3N-1}{2}=\frac{3}{2}N^2-\frac{1}{2}N.$$

During the first N rounds, all three children recite a total of $6+15+24+33+\cdots+(9N-3)$ numbers. The sum of this arithmetic series is

$$N\cdot\frac{9N+3}{2}=\frac{9}{2}N^2+\frac{3}{2}N.$$

The number of rounds prior to the round during which Tadd says his 2019th number is the greatest value of N such that $\frac{3}{2}N^2-\frac{1}{2}N<2019$.

Multiplying by 2 yields $3N^2 - N < 4038$. If $N = 36$, then the left side is $3 \cdot 1296 - 36 = 3852$; and if $N = 37$, then the left side is $3 \cdot 1369 - 37 = 4070$. Therefore the N being sought is 36, and Tadd has recited $\frac{1}{2} \cdot 3852 = 1926$ numbers after 36 rounds. During round 37, Tadd will recite $3 \cdot 37 - 2 = 109$ numbers, including his 2019th number.

After 36 rounds have been completed, the children combined will have recited

$$\frac{9}{2} \cdot 36^2 + \frac{3}{2} \cdot 36 = \frac{9}{2} \cdot 1296 + 54 = 5832 + 54 = 5886$$

numbers, the integers from 1 through 5886.

Tadd will say his 2019th number when he has completed reciting $2019 - 1926 = 93$ numbers in round 37. This number is $5886 + 93 = 5979$.

24. **Answer (B):** Because

$$x^3 - 22x^2 + 80x - 67 = (x-p)(x-q)(x-r),$$

multiplying the given equation by the common denominator yields

$$1 = A(s-q)(s-r) + B(s-p)(s-r) + C(s-p)(s-q).$$

This is now a polynomial identity that holds for infinitely many values of s, so it must hold for all s. This means the condition that $s \notin \{p, q, r\}$ can be removed.

Setting $s = p$ yields $1 = A(p-q)(p-r)$, so $\frac{1}{A} = (p-q)(p-r)$. Similarly, $\frac{1}{B} = (q-p)(q-r)$ and $\frac{1}{C} = (r-p)(r-q)$. Hence

$$\begin{aligned}\frac{1}{A} + \frac{1}{B} + \frac{1}{C} &= (p-q)(p-r) + (q-p)(q-r) + (r-p)(r-q) \\ &= (p+q+r)^2 - 3(pq+qr+rp).\end{aligned}$$

By Viète's Formulas $p+q+r$ is the negative of the coefficient of x^2 in the polynomial and $pq+qr+rp$ is the coefficient of the x term, so the requested value is $22^2 - 3 \cdot 80 = 244$. (The numerical values (p, q, r, A, B, C) are approximately $(1.23, 3.08, 17.7, 0.0329, -0.0371, 0.00416)$.)

25. **Answer (D):** Let

$$A_n = \frac{(n^2-1)!}{(n!)^n}.$$

First, note that A_n is an integer when $n = 1$. Next, observe that if n is prime, then A_n is not an integer because the numerator has $n-1$ factors of n but the denominator has n such factors. Note also that A_4 is not an integer, because the numerator, $15!$, has $7+3+1 = 11$ factors of 2, whereas the denominator, $(4!)^4$, has 12 factors of 2. Therefore for $n \geq 2$, in order for A_n to be an integer, a necessary condition is that n be composite and greater than 4. The following argument shows that this condition is also sufficient.

First note that
$$\frac{n!}{n^2} = \frac{(n-1)!}{n}.$$
If $n = ab$, where a and b are distinct positive integers greater than 1, then $\frac{(n-1)!}{n}$ is an integer because both a and b appear as factors in $(n-1)!$. Otherwise $n = p^2$ for some odd prime p. In this case $p^2 - 1 \geq 2p$, so $(n-1)!$ has at least two factors of p and again $\frac{(n-1)!}{n}$ is an integer.

Now the number
$$\frac{(n^2)!}{(n!)^{n+1}}$$
is an integer because this expression counts the number of ways to separate n^2 objects into n groups of size n without regard to the ordering of the groups (which accounts for the extra factor of $n!$ in the denominator).

By combining the previous two paragraphs, it follows that
$$A_n = \frac{(n^2-1)!}{(n!)^n} = \frac{(n^2)!}{(n!)^{n+1}} \cdot \frac{n!}{n^2}$$
is an integer if and only if $n = 1$ or n is composite and greater than 4. Thus the answer is 50 minus 1 minus the number of primes less than or equal to 50, which is $49 - 15 = 34$.

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